Special correspondences and Chow traces of Landweber-Novikov operations

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Abstract

We prove that the function field of a variety which possesses a special correspondence in the sense of M. Rost preserves rationality of cycles of small codimensions. This fact was proven by Vishik in the case of quadrics and played the crucial role in his construction of fields with u-invariant $2^r + 1$. The main technical tools are the algebraic cobordism of Levine-Morel, the generalised degree formula and the divisibility of Chow traces of certain Landweber-Novikov operations. As a direct application of our methods we prove the similar fact for all F_4-varieties.

MSC: 14C15, 55N22

1 Introduction

In paper [Vi07] A. Vishik using the techniques of symmetric operations in algebraic cobordism (see [Vi06]) proved that changing the base field by the function field of a smooth projective quadric doesn't change the property of being rational for cycles of small codimension. This fact which he calls the Main Tool Lemma plays the crucial role in his construction of fields with u-invariant $2^r + 1$.

In the present paper we prove the M.T.L. for a class of varieties introduced by M. Rost in the context of the Bloch-Kato conjecture. Namely, for varieties which possess a *special correspondence* (see [Ro06, Definition 5.1]).

As in Vishik's proof the main technical tools are the algebraic cobordism of M. Levine and F. Morel, the generalised degree formula and the divisibility of Chow traces of certain Landweber-Novikov operations. Therefore, we always assume that our base field k has characteristic 0.

We use the following notation. All smooth varieties are assumed to be irreducible. By $\overline{k}$ we denote the algebraic closure of k and by $X_{\overline{k}}$ the respective base change $X \times_k \overline{k}$ of a variety X. Given a prime p by $\overline{\mathrm{CH}}(X)$ we denote the Chow ring of X modulo its p-torsion part and by $\mathrm{Ch}(X) = \overline{\mathrm{CH}}(X) \otimes \mathbb{Z}/p$ the respective Chow ring with $\mathbb{Z}/p$-coefficients. The Chow ring is a graded ring. Its m-th graded component is given by cycles of codimension m and is denoted by $\mathrm{Ch}^m(X)$. We say that a cycle $y \in \mathrm{Ch}^m(X_{\overline{k}})$ is *defined over k* if it belongs to the image of the restriction map $\mathrm{res}_{\overline{k}/k} \colon \mathrm{Ch}^m(X) \to \mathrm{Ch}^m(X_{\overline{k}})$.

The following notion will be central in this paper

1.1 Definition. Let X be a smooth proper irreducible variety over a field k of dimension n, p be a prime and d be an integer $0 \le d \le n$. Assume that X has no zero-cycles of degree coprime to p. We say X is a *d-splitting variety mod p* if for any smooth quasi-projective variety Y over k, for any $m < d$ and for any cycle $y \in \mathrm{Ch}^m(Y_{\overline{k}})$ the following condition holds

$$y \text{ is defined over } k \iff y_{\overline{k}(X)} \text{ is defined over } k(X). \tag{1}$$

1.2 Example. Let Q be an anisotropic projective quadric over k of dimension $n > 2$ and $p = 2$. Then according to A. Vishik

(a) Q is a $[\frac{n+1}{2}]$-splitting variety [Vi07, Cor. 3.5.(1)];

(b) Q is a n-splitting variety if and only if Q possesses a Rost projector (the proof is unpublished).

The main result of the paper is the following generalisation of 1.2.(b)

1.3 Theorem. *Let X be a smooth proper irreducible variety of dimension n over a field of characteristic 0. Assume that X has no zero-cycles of degree coprime to p. If X possesses a special correspondence in the sense of Rost, then X is a $\frac{n}{p-1}$-splitting variety and the value $\frac{n}{p-1}$ is optimal.*

As an application of the techniques used in the proof of 1.3, we provide a complete list of d-splitting projective homogeneous varieties of type F_4.

1.4 Corollary. *Let X be a projective homogeneous variety of type F_4 (see Sect 4.IV) and p be one of its torsion primes (2 or 3). Assume that X has no zero-cycles of degree coprime to p. Then depending on p we have*

$p = 2$: If X is of type F_4/P_4, then X is a $(\dim X)$-splitting variety. For all other types X is a 3-splitting variety and this value is optimal.

$p = 3$: X is always a 4-splitting variety and this value is optimal.

1.5 Example. An example of a non-homogeneous variety which possesses a special correspondence for $p = 3$ was provided recently by N. Semenov (see [Se08]). By Theorem 1.3 it also provides an example of a 4-splitting variety.

2 Mod-p operations

In the present section we introduce certain operations

$$\phi_p^{q(t)} \colon \Omega(X) \to \mathrm{Ch}(X) \text{ parametrised by } q(t) \in \mathrm{Ch}(X)[[t]]$$

from the ring of algebraic cobordism $\Omega(X)$ to the Chow ring $\mathrm{Ch}(X)$ with $\mathbb{Z}/p$-coefficients of a smooth variety X, where p is a given prime. We also define the Rost number $\eta_p(X)$ and discuss its properties.

2.1. The group $\Omega^m(X)$ of cobordism cycles is generated by classes of proper morphisms $[Z \to X]$ of pure codimension m with Z smooth. There are cohomological operations on Ω parametrised by partitions called Landweber-Novikov operations and denoted by S_{LN}. These operations commute with pull-backs, satisfy projection and Cartan formulas. In our paper we will deal only with operations given by partitions $(p-1, p-1, \ldots, p-1)$. Such an operation will be denoted by S_{LN}^i, where i is the length of a partition.

There is a commutative diagram for any integer m

$$
\begin{array}{ccccc}
\Omega^m(X) & \xrightarrow{pr} & \overline{\mathrm{CH}}^m(X) & \longrightarrow & \mathrm{Ch}^m(X) \\
{\scriptstyle S_{LN}^i}\downarrow & & & & \downarrow{\scriptstyle S^i} \\
\Omega^{m+i(p-1)}(X) & \xrightarrow{pr} & \overline{\mathrm{CH}}^{m+i(p-1)}(X) & \longrightarrow & \mathrm{Ch}^{m+i(p-1)}(X)
\end{array}
$$

where S_{LN}^i is the Landweber-Novikov operation, $pr \colon \Omega(X) \to \mathrm{CH}(X)$ is the canonical morphism of oriented theories and S^i is the i-th reduced p-power operation.

By properties of reduced power operations $S^i = 0$ if $i > m$. By commutativity of the diagram it means that the composite $pr \circ S_{LN}^i$ is divisible by p in $\overline{\mathrm{CH}}^{m+i(p-1)}(X)$. Define (cf. [Vi06, 3.3])

$$\phi_p^{t^{(p-1)a}} = \tfrac{1}{p}(pr \circ S_{LN}^i) \mod p, \text{ where } a = i - m > 0. \tag{2}$$

If r is not divisible by $(p-1)$, then we set $\phi_p^{t^r} = 0$. Hence, we have constructed an operation $\phi_p^{t^r}$, $r > 0$, which maps $\Omega^m(X)$ to $\mathrm{Ch}^{r+pm}(X)$.

Finally, given a power series $q(t) \in \mathrm{Ch}(X)[[t]]$ define

$$\phi_p^{q(t)} = \sum_{r \geq 0} q_r \phi_p^{t^r}, \text{ where } q(t) = \sum_{r \geq 0} q_r t^r.$$

By the very definition operations $\phi_p^{q(t)}$ are additive and respect pull-backs.

2.2 Definition. Let $[U]$ be the class of a smooth projective variety U of dimension d in the Lazard ring $\mathbb{L}_d = \Omega^{-d}(k)$. Assume that $(p-1)$ divides d. Then the integer $\phi^{t^{dp}}([U]) \in \mathrm{Ch}^0(pt) = \mathbb{Z}/p$ will be called the *Rost number* of U and will be denoted by $\eta_p(U)$.

Using the definition of S_{LN} the number $\eta_p(U)$ can be computed as follows. Let $\xi_1, \xi_2, \ldots, \xi_d$ be the roots of the total Chern class of the tangent bundle of U. Define $c(T_U)^{(p)} = \prod_{j=1}^{d}(1 + \xi_j^{p-1})$. Then

$$\eta_p(U) = \tfrac{1}{p} \deg \left(c(T_U)^{(p)} \right)^{-1}. \tag{3}$$

Indeed, it coincides with the number $b_d(U)/p$ introduced in [Ro06, Sect. 9].

2.3 Lemma. *(cf. [Vi07, Prop.2.3]) Let U be a smooth projective variety of positive dimension d and $[U]$ be its class in the Lazard ring. Let $\beta \in \Omega^j(X)$. Then*

$$\phi_p^{t^r}([U] \cdot \beta) = \eta_p(U) \cdot pr(S_{LN}^i(\beta)), \text{ where } r = (p-1)(i-j) + dp > 0.$$

Observe that $\phi_p^{t^r}([U] \cdot \beta) = 0$ if d is not divisible by $(p-1)$.

Proof. Let $q \colon X \to \mathrm{Spec}\, k$ be the structure map. Then by Cartan formula

$$\tfrac{1}{p}(pr \circ S_{LN}^i)([U] \cdot \beta) = \sum_{\alpha+\beta=i} \tfrac{1}{p}pr(S_{LN}^\alpha(q^*[U])) \cdot pr(S_{LN}^\beta(\beta)).$$

To finish the proof observe that if $\alpha \neq \frac{d}{p-1}$, then $pr(S_{LN}^\alpha(q^*[U])) = q^*(pr \circ S_{LN}^\alpha([U])) = 0$ in $\overline{\mathrm{CH}}(X)$ by dimension reasons. $\square$

2.4 Corollary. *(cf. [LM07, Lemma 4.4.20]) Let U and V be smooth projective varieties of positive dimensions. Then $\eta_p(U \cdot V) = 0$ in $\mathbb{Z}/p$.*

Proof. Apply Lemma 2.3 to $X = pt$ and $\beta = [V]$. Then $\eta_p(U \cdot V) = \eta_p(U) \cdot pr(S^i_{LN}([V]))$, where the last factor is divisible by p and, hence, becomes trivial modulo p. □

According to [LM07, Remark 4.5.6] the kernel of the canonical morphism $pr \colon \Omega(X) \to \mathrm{Ch}(X)$ is generated by classes of positive dimensions, i.e. $ker(pr) = \mathbb{L}_{>0} \cdot \Omega(X)$. Hence, any $\gamma \in ker(pr)$ can be written as

$$\gamma = \sum_{u_Z \in \mathbb{L}_{>0}} u_Z \cdot [Z \to X]. \tag{4}$$

Let $\gamma_{pt} \in \mathbb{L}$ denote the class u_Z corresponding to the point $Z = pt$.

2.5 Lemma. *Let $\pi \in \Omega(X \times X)$ be an idempotent and $\gamma \in \Omega(X)$ be such that $pr(\pi_\star(\gamma) - \gamma) = 0$, where $\pi_\star$ is the realization. Then $\eta_p(\pi_\star(\gamma)_{pt}) = \eta_p(\gamma_{pt})$.*

Proof. In presentation (4) let $\pi_\star(\gamma) - \gamma = \sum_{u_Z \in \mathbb{L}_{>0}} u_Z \cdot [Z \to X]$. Since π is an idempotent and $\pi_\star$, q_* are $\mathbb{L}$-module homomorphisms, we obtain

$$0 = q_* \pi_\star(\pi_\star(\gamma) - \gamma) = \sum_{u_Z \in \mathbb{L}_{>0}} u_Z \cdot q_* \pi_\star([Z \to X]).$$

Apply η_p to the both sides of the equality. By Corollary 2.4 all summands with $\dim Z > 0$ become trivial (modulo p). Hence, $\eta_p(u_{pt}) = 0 \mod p$. □

2.6 Corollary. *Let π and γ be as above. Then $\eta_p(q_*(\pi_\star(\gamma) - \gamma)) = 0$, where $q \colon X \to Spec\, k$ is the structure map.*

Proof. Observe that $\eta_p(q_*(\pi_\star(\gamma) - \gamma)) = \sum_{u_Z \in \mathbb{L}_{>0}} \eta_p(u_Z \cdot [Z])$, where all summands with $\dim Z > 0$ are trivial by Corollary 2.4 and the summand with $Z = pt$ is trivial by Lemma 2.5. □

The next important lemma is a direct consequence of the result by M. Rost [Ro06, Lemma 9.3].

2.7 Lemma. *Let X be a variety which possesses a special correspondence. Then for any $\gamma \in \Omega_{>0}(X)$ the $\deg pr(S^i_{LN}(\gamma))$ is divisible by p in $\overline{\mathrm{CH}}(X)$.*

Proof. We have $S^\bullet = S_\bullet \cdot c_\Omega(-T_{X_{\overline{k}}})$, where $(S^\bullet)$ $S_\bullet$ are the (co-)homological operations. Hence, $\deg pr(S^i_{LN}(\gamma)) = \deg\big(pr(S_\bullet(\gamma)) \cdot pr(c_\Omega(-T_{X_{\overline{k}}}))\big)$. Since all Chern classes of the tangent bundle of $X_{\overline{k}}$ are defined over k and X possesses a special correspondence, according to [Ro06, Lemma 9.3] we obtain that

$$\deg\big(pr(S_\bullet(\gamma)) \cdot pr(c_\Omega(-T_{X_{\overline{k}}}))\big) = \deg pr(S_i(\gamma)) \mod p.$$

Since $S_\bullet$ respect push-forwards and γ has positive dimension, $\deg pr(S_i(\gamma))$ is divisible by p as well. $\qquad\square$

3 Construction of a cycle defined over k

In the present section we prove Theorem 1.3. The proof consists of several steps. First, following Vishik's arguments for a given $y \in \mathrm{Ch}^m(X_{\overline{k}})$ we construct a cycle $\bar\omega$ defined over k in the cobordism ring of the product $\Omega(X_{\overline{k}} \times Y_{\overline{k}})$. To do this we essentially use the surjectivity of the canonical map $pr\colon \Omega \to \mathrm{CH}$. The motivic decomposition of X provides an idempotent cycle π. Applying the realization of π to $\bar\omega$ we obtain a cycle ρ defined over k which can be written in the form (5). To finish the proof we apply two operations $\phi_p^{tr} \circ p_{Y*}$ and $p_{Y*} \circ \phi_p^{tr'}$ to the cycle ρ. The direct computations which are based on the generalised degree formula and [Ro06, Lemma 9.3] show that the difference $(\phi_p^{tr} \circ p_{Y*} - p_{Y*} \circ \phi_p^{tr'})(\rho)$ is defined over k and provides the cycle y.

I. We start as in the proof of [Vi07, Thm. 3.1]. Let Y be a smooth quasi-projective variety over k. Let $y \in \mathrm{Ch}^m(Y_{\overline{k}})$ be such that $y_{\overline{k}(X)}$ is defined over $k(X)$. We want to show that y is defined over k for all $m < d$.

Consider the commutative diagram

$$
\begin{array}{ccccccc}
\omega & \Omega^m(X \times Y) & \xrightarrow{pr} & \mathrm{Ch}^m(X \times Y) & \xrightarrow{pr_1^*} & \mathrm{Ch}^m(Y_{k(X)}) & u \\
\Big\downarrow & \ \ \Big\downarrow{\scriptstyle res} & & \Big\downarrow{\scriptstyle res} & & \Big\downarrow{\scriptstyle res} & \Big\downarrow \\
\bar\omega & \Omega^m(X_{\overline{k}} \times Y_{\overline{k}}) & \xrightarrow{pr} & \mathrm{Ch}^m(X_{\overline{k}} \times Y_{\overline{k}}) & \xrightarrow{\bar{pr}_1^*} & \mathrm{Ch}^m(Y_{\overline{k}(X)}) & y_{\overline{k}(X)}
\end{array}
$$

where the pull-back pr_1^* is surjective by the localisation sequence and pr is surjective due to [LM07, Thm.4.5.1]. By the hypothesis there exists a preimage u of $y_{\overline{k}(X)}$ by means of res. By the surjectivity of pr and pr_1^*, there exists a preimage ω of u. Set $\bar\omega = \mathrm{res}(\omega)$.

II. Let X be a variety which possesses a special correspondence and has no zero-cycles of degree coprime to p. By the results of M. Rost it follows that

(a) $\eta_p(X_{\bar{k}}) \neq 0 \mod p$ (see [Ro06, Thm. 9.9]),

(b) $n = p^s - 1$ (see [Ro06, Cor. 9.12]),

(c) the Chow motive of X contains an indecomposable summand M which over $\bar{k}$ splits as a direct sum of Tate motives twisted by the multiples of $d = \frac{n}{p-1}$ (see [Ro06, Prop. 7.14])

$$M_{\bar{k}} \simeq \bigoplus_{i=0}^{p-1} \mathbb{Z}/p\{di\}.$$

Let π be an idempotent defining the respective Ω-motive M. Then the realization $\rho = \pi_\star(\bar{\omega})$ is defined over k and can be written as (cf. [Vi07, p.368])

$$\rho = x_n \times y_n + \sum_{i=1}^{p-2} x_{di} \times y_{di} + x_0 \times y_0 \in \Omega^m(X_{\bar{k}} \times Y_{\bar{k}}), \qquad (5)$$

where $x_j \in \Omega_j(X_{\bar{k}})$, $y_j \in \Omega^{m-n+j}(Y_{\bar{k}})$, $x_0 = [pt \hookrightarrow X_{\bar{k}}]$, $x_n = \pi_\star(1)$ and $pr(y_n) = y$ (cf. [Vi07, Lemma 3.2]).

III. Let p_X^* and p_Y^* denote the pull-backs induced by projections $X \times Y \to X, Y$. Since $m < d$, $r = (dp - m)(p-1) > 0$. Consider the cycle $\phi_p^{t^r}(p_{Y*}(\rho))$. It is defined over k and has codimension m.

3.1 Lemma. $\phi_p^{t^r}(p_{Y*}(\rho)) = \eta_p(X_{\bar{k}}) \cdot y + \phi_p^{t^r}(y_0)$ in $\mathrm{Ch}^m(Y_{\bar{k}})$.

Proof. By the projection formula

$$\phi_p^{t^r}(p_{Y*}(x_j \times y_j)) = \phi_p^{t^r}(p_{Y*}(p_X^*(x_j) \cdot p_Y^*(y_j))) = \phi_p^{t^r}(q_*(x_j) \cdot y_j),$$

where $x_j \times y_j$ is a summand of (5) and $q \colon X \to Spec\,k$ is the structure map. Assume $0 < j < n$. By Lemma 2.3 we obtain

$$\phi_p^{t^r}(p_{Y*}(x_j \times y_j)) = \eta_p(q_*(x_j)) \cdot pr(S_{LN}^l(y_j)), \text{ where } l = \frac{n-j}{p-1}.$$

Since $codim(y_j) = m - n + j \leq m - n + (p-2)d = m - d < 0$, we have $pr(S^l_{LN}(y_i)) = S^l(pr(y_j)) = 0$ in $\mathrm{Ch}^m(X_{\overline{k}})$. Therefore, only the very right and the left summands of (5) remain non-trivial after applying $\phi^{t^r}_p \circ p_{Y*}$.

Now by Lemma 2.6 the first summand is equal to

$$\phi^{t^r}_p(p_{Y*}(x_n \times y_n)) = \eta_p(q_*(x_n)) \cdot pr(y_n) = \eta_p(q_*(\pi_*(1))) \cdot y = \eta_p(X_{\overline{k}}) \cdot y.$$

and the last summand $\phi^{t^r}_p(p_{Y*}(x_0 \times y_0)) = \phi^{t^r}(q_*(x_0) \cdot y_0) = \phi^{t^r}_p(y_0)$. $\qquad \square$

Since $m < d$, $r' = (d-m)(p-1) > 0$. Consider the cycle $p_{Y*}(\phi^{t^{r'}}_p(\rho))$. It is defined over k and has codimension m.

3.2 Lemma. $p_{Y*}(\phi^{t^{r'}}_p(\rho)) = \phi^{t^r}_p(y_0)$ *in* $\mathrm{Ch}^m(Y_{\overline{k}})$.

Proof. By the very definition $p_{Y*}\big(\phi^{t^{r'}}_p(x_j \times y_j)\big) = \frac{1}{p}p_{Y*}pr(S^d_{LN}(x_j \times y_j))$. By the projection and Cartan formulas the latter can be written as

$$\frac{1}{p}\deg\big(pr(S^a_{LN}(x_j))\big) \cdot pr(S^{d-a}_{LN}(y_j)), \text{ where } a = \frac{j}{p-1}.$$

Since $m < d$, the cycles y_j have negative codimensions for all $j < n$ and, therefore, $pr(S^{d-a}_{LN}(y_j))$ is divisible by p for all $j < n$. On the other hand, by Lemma 2.7 the $\deg\big(pr(S^a_{LN}(x_j))\big)$ is divisible by p for all $j > 0$. Hence, $p_{Y*}\big(\phi^{t^{r'}}_p(x_j \times y_j)\big) = 0$ for all $0 < j < n$.

Consider the case $j = n$. Recall that $x_n = \pi_*(1)$. Then we have

$$\frac{1}{p}\deg\big(pr(S^d_{LN}(x_n))\big) = \sum_{u_Z \in \mathbb{L}_{>0}} \eta_p(u_Z) \cdot \deg pr(S^{d-a}_{LN}([Z \to X])),$$

where $a = \frac{\dim Z}{p-1}$ and $x_n - 1 = \sum u_Z[Z \to X]$ in presentation (4). Observe that it is trivial mod p, since for all $\dim Z > 0$ the degree of the cycle $pr(S^{d-a}_{LN}([Z \to X]))$ is trivial by Lemma 2.7 and for $\dim Z = 0$ the Rost number $\eta_p(u_{pt})$ is trivial by Lemma 2.5.

Hence, only the very last summand, i.e. $x_0 \times y_0$, remains non-trivial after applying $p_{Y*} \circ \phi^{t^{r'}}$ which gives $p_{Y*}(\phi^{t^{r'}}(x_0 \times y_0)) = \phi^{t^r}(y_0)$. $\qquad \square$

IV. By Lemmas 3.1 and 3.2 the following cycle is defined over k

$$\phi^{t^r}_p(p_{Y*}(\rho)) - p_{Y*}(\phi^{t^{r'}}_p(\rho)) = \eta_p(X_{\overline{k}}) \cdot y.$$

Since $\eta_p(X_{\overline{k}}) \neq 0 \mod p$, the cycle y is defined over k, therefore, X is a d-splitting variety.

To see that $d = \frac{n}{p-1}$ is an optimal value take $Y = X$ and consider the cycle $y \in \mathrm{Ch}^d(X_{\overline{k}})$ which generates the Chow group of the Tate motive $\mathbb{Z}/p\{n-d\}$ in the decomposition of $M_{\overline{k}}$ over $\overline{k}$. Observe that y coincides with the cycle H introduced in [Ro06, Sect. 5]. Since M splits over $k(X)$, $y_{\overline{k}(X)}$ is defined over $k(X)$. By condition (1) we obtain that y is defined over k, i.e. M splits over k which contradicts to the indecomposability of M. The theorem is proven.

4 F_4-varieties

In the present section we apply our methods to describe all d-splitting varieties of type F_4.

I. Let X be a smooth geometrically cellular variety over k of dimension n. As in the beginning of the previous section given a cycle $y \in \mathrm{Ch}^m(Y_{\overline{k}})$, where Y is smooth quasi-projective, we construct a cobordism cycle $\bar{\omega} \in \Omega^m(X_{\overline{k}} \times Y_{\overline{k}})$ defined over k.

II. Assume that the motive of X contains a motive $M = (X, \pi)$ such that

$$\pi_{\overline{k}} = \gamma \times \gamma^{\vee} + \sum_{i=0}^{p-2} x_{di} \times x_{di}^{\vee}, \text{ where } d > 1,$$

the cycle $\gamma \in \Omega_{(p-1)d}(X_{\overline{k}})$ is defined over k, $\gamma^{\vee}$ denotes its Poincare dual, i.e. $pr(\gamma \cdot \gamma^{\vee}) = pt$, and $x_j \in \Omega_j(X_{\overline{k}})$. Then the realization $\rho = \pi_{\star}(p_X^*(\gamma) \cdot \bar{\omega}) \in \Omega^{m+n-g}(X_{\overline{k}} \times Y_{\overline{k}})$ is defined over k and can be written as (cf. (5))

$$\rho = x_g \times y_g + \sum_{i=1}^{p-2} x_{di} \times y_{di} + x_0 \times y_0,$$

where $g = (p-1)d$, $x_g = \pi_{\star}(\gamma)$ and $pr(y_g) = y$.

The transposed cycle π^t defines an opposite direct summand $M^t = (X, \pi^t)$ of the motive of X (the one which contains the generic point of X over $\overline{k}$).

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The realization $\rho' = \pi_\star^t(\bar\omega)$ is defined over k and can be written as

$$\rho' = x^{(0)} \times y^{(0)} + \sum_{i=1}^{p-2} x^{(di)} \times y^{(di)} + x^{(g)} \times y^{(g)} \in \Omega^m(X_{\overline{k}} \times Y_{\overline{k}}),$$

where $x^{(j)} \in \Omega^j(X_{\overline{k}})$ and $y^{(g)} = y_0$.

III. Since the Chow group of the cellular variety $X_{\overline{k}}$ is torsion-free, we may use Mod-p operations over $\overline{k}$. We now apply the operations $\phi^{t^r} \circ p_{Y*}$ and $p_{Y*} \circ (p_X^*(\gamma) \cdot \phi^{t^{r'}})$ to the cycles ρ and ρ' respectively.

Applying the first operation and repeating the arguments of the proof of Lemma 3.1 we obtain that for any $m < d$, $r = (dp - m)(p - 1)$, the following cycle is defined over k

$$\phi_p^{t^r}(p_{Y*}(\rho)) = \eta_p(\gamma) \cdot y + \phi_p^{t^r}(y_0) \text{ in } \text{Ch}^m(Y_{\overline{k}}).$$

To apply the second operation for any $m < d$ consider the cycle $\delta = p_{Y*}(p_X^*(\gamma) \cdot \phi_p^{t^{r'}}(\rho'))$, where $r' = (d - m)(p - 1)$. It is defined over k and has codimension m. Since we don't have the version of [Ro06, Lemma 9.3] for an arbitrary variety, to compute δ we have to treat each torsion prime case separately.

For $p = 2$ we obtain $\delta = \phi_p^{t^r}(y_0)$ by dimension reasons. Indeed, in this case the cycle ρ' consists only of two summands

$$\rho' = x^{(0)} \times y^{(0)} + \gamma^\vee \times y^{(g)},$$

where the first summand vanishes, since $S_{LN}^\alpha(x^{(0)}) = 0$ if $|\alpha| > 0$ and the second summand gives the required cycle $\phi_p^{t^r}(y_0)$.

For $p = 3$ the cycle ρ' consists of three terms

$$\rho' = x^{(0)} \times y^{(0)} + x^{(d)} \times y^{(d)} + \gamma^\vee \times y^{(g)},$$

where again the first summand vanishes, the last gives $\phi^{t^r}(y_0)$ and the middle gives

$$\deg\left(pr(\gamma) \cdot pr(S_{LN}^\alpha(x^{(d)}))\right) \cdot \tfrac{1}{p} pr(S_{LN}^\beta(y^{(d)})), \text{ where } \alpha = \beta = d/2. \qquad (6)$$

Hence, following the part IV of the previous section to prove that X is a d-splitting variety for $p = 2$ or 3 it is enough to assume that $\eta_p(\gamma) \neq 0$ mod p and that (6) vanishes for $p = 3$.

IV. We use the following notation. Let G be a simple linear algebraic group over k. We say a projective homogeneous G-variety X is of type $\mathcal{D}$, if the group $G_{\overline{k}}$ has a root system of type $\mathcal{D}$. Moreover, if $X_{\overline{k}}$ is the variety of parabolic subgroups of $G_{\overline{k}}$ defined by the subset of simple roots S of $\mathcal{D}$, then we say that X is of type $\mathcal{D}/P_S$. In this notation $P_{\mathcal{D}}$ defines a Borel subgroup and P_i, $i \in \mathcal{D}$, defines a maximal parabolic subgroup (our enumeration of roots follows Bourbaki).

Given an F_4-variety X we provide a cycle γ satisfying $\eta_p(\gamma) \neq 0 \mod p$ as follows

$p = 2$: If X is generically split over the 2-primary closure of k, then we may assume that X is of type F_4/P_1. In this case X has dimension 15 and by the main result of [PSZ] the Chow motive of X with $\mathbb{Z}/p$-coefficients splits as a direct sum of twisted copies of a certain motive $M = (X, \pi)$ with the generating function $P(M_{\overline{k}}, t) = 1 + t^3$ (see p.33 of [PSZ]). Since the Chow group $\mathrm{Ch}_r(X_{\overline{k}})$ has rank 1 for $r = 0 \ldots 3$, the idempotent $\pi_{\overline{k}}$ can be written as $\pi_{\overline{k}} = \gamma \times \gamma^{\vee} + pt \times 1$, where γ is represented by a 3-dimensional subquadric $Q_3 \hookrightarrow X_{\overline{k}}$ which is an additive generator of $\mathrm{Ch}_3(X_{\overline{k}})$ defined over k. Since $\eta_2(Q_3) = 1$, X is a d-splitting variety with $d = 3$.

If X is not generically split, i.e. X is of the type F_4/P_4, then X is a splitting variety of the symbol given by the cohomological invariant f_5. By the result of Rost [Ro06, Rem. 2.3 and § 8] X is a variety which possesses a special correspondence. Hence, by Thm. 1.3 X is a 15-splitting variety.

$p = 3$: In this case all F_4-varieties are generically split over the 3-primary closure of k and we may assume that X is of type F_4/P_4. Similar to the previous case using the motivic decomposition from [PSZ] we obtain an idempotent

$$\pi_{\overline{k}} = \gamma \times \gamma^{\vee} + x_4 \times x_4^{\vee} + pt \times 1,$$

where $\gamma \in \Omega_8(X_{\overline{k}})$ is defined over k and $x_4 \in \Omega_4(X_{\overline{k}})$ is not. By the explicit formulae from [NSZ, 5.5] we may identify $pr(\gamma)$ with the 7-th power of the generator H of the Picard group of $X_{\overline{k}}$. which is the only cycle in $\mathrm{Ch}_8(X_{\overline{k}})$ defined over k. Since H is very ample, H^7 can be represented by a smooth projective subvariety Z of $X_{\overline{k}}$. Hence, we may identify γ with the class $[Z \hookrightarrow X_{\overline{k}}]$. Then the direct computations using the adjunction formula [Fu98, Example 3.2.12] show that $\eta_3(Z) \neq 0 \mod 3$.

To prove the vanishing of the cycle (6) it is enough to prove the vanishing of the cycle

$$\deg\left(pr(\gamma) \cdot S^2(pr(x^{(4)}))\right) \cdot \phi_3^{t^{12-2m}}(y^{(4)}).$$

Direct computations show that $S^2(\mathrm{Ch}^4(X_{\overline{k}}))$ is trivial, hence, $\delta = \phi_p^{tr}(y_0)$ and X is a d-splitting variety with $d = 4$. The Corollary 1.4 is proven.

4.1 Remark. Let X be a d-splitting geometrically cellular variety. As an immediate consequence of [KM06, Cor. 4.11] we obtain the following bound for a canonical p-dimension of X

$$cd_p(X) \geq d.$$

In the case of a variety of type F_4/P_4 it gives $cd_2(X) = \dim X = 15$.

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